THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5220 Complex Analysis and its Applications 2016-2017
Suggested Solution to Assignment 5

1 Given that $f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$ with radius of convergence R around z_0. We want to show that

$$f^{(n)}(z) = \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} a_{n+k}(z-z_0)^k \text{ for } n = 0, 1, 2, \dots$$

For $n = 0$, the statement is trivial. Assume that the statement is true for $n = m$. For $n = m+1$, by assumption we have

$$f^{(m)}(z) = \sum_{k=0}^{\infty} \frac{(m+k)!}{k!} a_{m+k}(z-z_0)^k$$

Differentiate both sides with respect to z, we have

$$\begin{aligned} f^{(m+1)}(z) =& \sum_{k=0}^{\infty} \frac{(m+k)!}{k!}(k) a_{m+k}(z-z_0)^{k-1} \\ =& \sum_{k=1}^{\infty} \frac{(m+k)!}{(k-1)!} a_{m+k}(z-z_0)^{k-1} \\ =& \sum_{k=0}^{\infty} \frac{(m+1+k)!}{k!} a_{m+1+k}(z-z_0)^{k} \end{aligned}$$

Hence the statement is true for $n = m+1$. By M.I., the statement is true for $n = 0, 1, 2, \dots$.

2 Note that for $0 < |z| < 1$,

$$\begin{aligned} \frac{e^z}{z(z^2+1)} =& \left(\frac{1}{z}\right)(e^z)\left(\frac{1}{1-(-z^2)}\right) \\ =& \left(\frac{1}{z}\right)(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \dots)\left(1 - z^2 + z^4 - z^6 + \dots\right) \\ =& \frac{1}{z}\left((1)(1) + (z)(1) + (\frac{z^2}{2}(1) + (1)(-z^2)) + (\frac{z^3}{6}(1) + (z)(-z^2)) + \dots\right) \\ =& \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + \dots \end{aligned}$$

3 Given $f(z) = 1 - \cos z$. Since we have $f(0) = 1 - \cos 0 = 0$, $f'(0) = \sin(0) = 0$ and $f''(0) = \cos(0) = 1 \neq 0$, f has a zero of order 2 at $z = 0$.

4 Since $f(z)$ has a zero at z_1 of order m_1, we know that there exists an analytic function $g_1(z)$ on D such that

$$f(z) = (z-z_1)^{m_1} g_1(z) \text{ and } g_1(z_1) \neq 0$$

Now since $f(z_2) = 0$ and $f(z) = (z - z_1)^{m_1} g_1(z)$, we must have $g_1(z_2) = 0$ and z_2 is a zero of $g_1(z)$ of order m_2. So there exists another analytic function $g_2(z)$ on D such that

$$g_1(z) = (z - z_2)^{m_2} g_2(z) \text{ and } g_2(z_2) \neq 0$$

By repeating the arguments several times and substituting the functions into $f(z)$, we can find an analytic function $g(z) = g_n(z)$ such that

$$f(z) = (z - z_1)^{m_1}(z - z_2)^{m_2} \ldots (z - z_n)^{m_n} g(z)$$

5 Since $f(z)$ is entire, it has a Talyor's series expansion $f(z) = \sum_{k=0}^{\infty} b_k z^k$. By assumption, for any real number x, we have $f(x) = \sum_{k=0}^{\infty} a_k x^k$. In particular, this power series converges for any $x \in \mathbb{R}$. Now define another function $g(z) = \sum_{k=0}^{\infty} a_k z^k$. Since $f(x)$ is a convergent power series for any $x \in \mathbb{R}$, $g(z)$ is an entire function. Note that $f(z)$ and $g(z)$ are both entire and $(f - g)(x) = 0$ for any $x \in \mathbb{R}$. That means the zeros of the entire function $(f - g)$ are not isolated. Hence we have $f(z) = g(z)$ for all $z \in \mathbb{C}$.

6 (a) See Q.7 in the suggested solution for assignment 2.

(b) ($\Longrightarrow$) By a) and the assumption, we have $g(z) = \overline{f(\overline{z})} = f(z)$ for any $z \in D$. In particular, we have

$$\overline{f(\overline{x})} = \overline{f(x)} = f(x) \text{ for any } x \in (a, b)$$

This shows that $f(x) \in \mathbb{R}$.

($\Longleftarrow$) Suppose $f(x) \in \mathbb{R}$ for any $x \in (a, b)$. Then $g(x) = \overline{f(\overline{x})} = \overline{f(x)} = f(x)$ for any $x \in (a, b)$. This implies $f(z) = g(z)$ for all $z \in D$ by uniqueness of analytic function.